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Barrier penetration for supersymmetric shape-invariant potentials

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Abstract. Exact reflection and transmission coefficients for supersymmetric shape-invariant potentials barriers are calculated by an analytical continuation of the asymptotic wavefunctions obtained via the introduction of new generalized ladder operators. The general form of the wavefunction is obtained by the use of the $\boldsymbol{F}(-\infty, +\infty)$-matrix formalism of Fröman and Fröman which is related to the evolution of asymptotic wavefunction coefficients.

1. Introduction

Quantum tunnelling through a potential barrier governs many interesting phenomena in physics ranging from fusion reactions in stars [1] to the study of transitions from metastable states [2]. There are very few exactly solvable examples of barrier penetration. Supersymmetric quantum mechanics has been shown to be a useful technique to explore exactly solvable problems in quantum mechanics (for a recent review see [3]). An integrability condition called shape invariance was introduced by Gendenshtein [4] and was cast into an algebraic form by Balantekin [5]. Reflection and transmission coefficients for a large class of shape-invariant potentials were given by Cooper *et al* [6]. A general operator method for calculating scattering amplitudes for supersymmetric shape-invariant potentials was introduced by Khare and Sukhatme [7]. Even though an approximate method in the context of the supersymmetric semiclassical approximation [8] to calculate tunnelling through one-dimensional potential barriers was presented in [9], exact tunnelling probabilities for shape-invariant barriers were not derived explicitly. We cover the latter subject in this paper.

Introducing the superpotential function

$$W(x) \equiv -\frac{\hbar}{\sqrt{2m}}\left[\frac{\Psi_0'(x)}{\Psi_0(x)}\right] \tag{1.1}$$

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where $\Psi_0(x)$ is the ground-state wavefunction of the Hamiltonian $\hat{H}$, and defining the operators

$$\hat{A} \equiv W(x) + \frac{\mathrm{i}}{\sqrt{2m}}\hat{p} \tag{1.2}$$

$$\hat{A}^\dagger \equiv W(x) - \frac{\mathrm{i}}{\sqrt{2m}}\hat{p} \tag{1.3}$$

we can show that

$$\hat{H} - E_0 = \hat{A}^\dagger \hat{A}. \tag{1.4}$$

Since the ground-state wavefunction satisfies the condition

$$\hat{A}\Psi_0(x) = 0 \tag{1.5}$$

the supersymmetric partner potentials

$$\hat{H}_1 = \hat{A}^\dagger \hat{A} \qquad \hat{H}_2 = \hat{A}\hat{A}^\dagger \tag{1.6}$$

have the same energy spectra except for the ground state of $\hat{H}_1$ which has no corresponding state in the spectra of $\hat{H}_2$. The corresponding potentials are given by

$$V_1(x) = [W(x)]^2 - \frac{\hbar}{\sqrt{2m}}\frac{\mathrm{d}W}{\mathrm{d}x} \tag{1.7}$$

$$V_2(x) = [W(x)]^2 + \frac{\hbar}{\sqrt{2m}}\frac{\mathrm{d}W}{\mathrm{d}x}. \tag{1.8}$$

The shape-invariance condition [4]

$$V_2(x, a_1) = V_1(x, a_2) + R(a_1) \tag{1.9}$$

can also be written as [5]

$$\hat{A}(a_1)\hat{A}^\dagger(a_1) = \hat{A}^\dagger(a_2)\hat{A}(a_2) + R(a_1) \tag{1.10}$$

where $a_{1,2}$ are a set of parameters that specify space-independent properties of the potentials (such as strength, range and diffuseness). The parameter a_2 is a function of a_1 and the remainder $R(a_1)$ is independent of $\hat{x}$ and $\hat{p}$. Not all exactly solvable potentials are shape invariant [10]. In the cases studied so far the parameters a_1 and a_2 are either related by a translation [10, 11] or a scaling [12]. Introducing the similarity transformation that replaces a_1 with a_2 in a given operator

$$\hat{T}(a_1)\, \hat{O}(a_1)\, \hat{T}^\dagger(a_1) = \hat{O}(a_2) \tag{1.11}$$

and the operators

$$\hat{B}_+ = \hat{A}^\dagger(a_1)\hat{T}(a_1) \tag{1.12}$$

$$\hat{B}_- = \hat{B}_+^\dagger = \hat{T}^\dagger(a_1)\hat{A}(a_1) \tag{1.13}$$

the Hamiltonian takes the form

$$\hat{H} - E_0 = \hat{B}_+\hat{B}_-. \tag{1.14}$$

Using equation (1.10) one can easily prove the commutation relation

$$[\hat{B}_-, \hat{B}_+] = \hat{T}^\dagger(a_1)R(a_1)\hat{T}(a_1) \equiv R(a_0) \tag{1.15}$$

where we have used an equality which follows from equation (1.11):

$$R(a_n) = \hat{T}(a_1)\, R(a_{n-1})\, \hat{T}^\dagger(a_1) \tag{1.16}$$

valid for any n. Equation (1.15) suggests that $\hat{B}_+$ and $\hat{B}_-$ are the appropriate creation and annihilation operators provided that their non-commutativity with $R(a_1)$ is taken into account. In this paper we extend the use of $\hat{B}_+$ and $\hat{B}_-$ operators for calculating the asymptotic behaviour of the wavefunctions related to incidence of a particle on a supersymmetric shape-invariant potential barrier and obtain the exact transmission and reflection coefficients.

2. Exact wavefunctions

For the tunnelling problem we consider Schrödinger equations of the form

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2} - V_1(x)\Psi(x) = (E-c)\Psi(x) \tag{2.1}$$

where c is, in general, a complex constant to be determined by inserting the appropriate superpotential into equation (1.7). The constant c in the continuum problem plays the role of the ground-state energy in the bound-state problem (cf equation (1.4)).

The wavefunctions for all currently known supersymmetric shape-invariant potential barriers can be calculated analytically using supersymmetric operator techniques [13, 14]. The final result can be expressed by single operators $\hat{B}_+$ and $\hat{B}_-^{-1}$ [15] or by pairs of these operators. In the first case we can use the two additional commutation relations

$$[\hat{B}_+\hat{B}_-, \hat{B}_+^n] = \sum_{k=1}^{n} R(a_k)\, \hat{B}_+^n \tag{2.2}$$

and

$$[\hat{B}_+\hat{B}_-, \hat{B}_-^{-n}] = \sum_{k=1}^{n} R(a_k)\, \hat{B}_-^{-n} \tag{2.3}$$

obtained by induction using the relations

$$R(a_n)\hat{B}_+ = \hat{B}_+ R(a_{n-1}) \tag{2.4}$$

$$R(a_n)\hat{B}_- = \hat{B}_- R(a_{n+1}) \tag{2.5}$$

that readily follow from equations (1.11)–(1.13). Considering that the Schrödinger equation can be written as

$$\hat{B}_+\hat{B}_-\Psi(x) = \Lambda\Psi(x) \tag{2.6}$$

then equations (2.2) and (2.3) imply that $\hat{B}_+$ and $\hat{B}_-$ can be used as ladder operators to solve equation (2.6) [5, 16]. To this end we introduce $\Psi_-^{(0)}(x)$ as the solution of the equation

$$\hat{A}_-(a_1)\, \Psi_-^{(0)}(x) = 0 = \hat{B}_-(a_1)\, \Psi_-^{(0)}(x) \tag{2.7}$$

which implies

$$\Psi_-^{(0)}(x, a_1) \sim \exp\left(-\frac{\sqrt{2m}}{\hbar}\int^x \mathrm{d}\xi\; W(\xi, a_1)\right). \tag{2.8}$$

If the function

$$f(n) = \sum_{k=1}^{n} R(a_k) \tag{2.9}$$

can be analytically continued so that the condition

$$f(\mu) = \Lambda \tag{2.10}$$

is satisfied for a particular (in general complex) value of μ, then equation (2.2) implies that one possible form for the solution of equation (2.6) is $\hat{B}_+^{\mu}\Psi_-^{(0)}(x, a_1)$. Similarly, if $\Psi_+^{(0)}(x)$ satisfies the equation

$$\hat{B}_+(a_1)\, \Psi_+^{(0)}(x) = 0 \tag{2.11}$$

which implies that

$$\hat{T}(a_1)\Psi_+^{(0)}(x) \sim \exp\left(\frac{\sqrt{2m}}{\hbar}\int^x \mathrm{d}\xi\, W(\xi, a_1)\right) \tag{2.12}$$

or

$$\Psi_+^{(0)}(x, a_0) \sim \exp\left(\frac{\sqrt{2m}}{\hbar}\int^x \mathrm{d}\xi\, W(\xi, a_0)\right) \tag{2.13}$$

then equation (2.3) implies that another possible form for the solution of equation (2.6) is $\hat{B}_-^{-\mu-1}\Psi_+^{(0)}(x, a_0)$. At this point we conclude that the components of the wavefunctions, written in terms of the singles operators $\hat{B}_+$ and $\hat{B}_-^{-1}$, for supersymmetric shape-invariant potential barriers can be written down as

$$\Psi_-(x) = \beta \hat{B}_+^{\mu}\Psi_-^{(0)}(x, a_1) \tag{2.14a}$$

$$\Psi_+(x) = \gamma \hat{B}_-^{-\mu-1}\Psi_+^{(0)}(x, a_0) \tag{2.14b}$$

where μ is obtained by the relation

$$\Lambda = \sum_{k=1}^{\mu} R(a_k) \tag{2.15}$$

where β and γ are constants and

$$\Psi_\pm^{(0)}(x, a_\mu) = \exp\left(\pm\frac{\sqrt{2m}}{\hbar}\int^x \mathrm{d}\xi\, W(\xi, a_\mu)\right). \tag{2.16}$$

With each value of μ we obtain several possible expressions for the components $\Psi_\pm(x)$ and the general expression for the wavefunction can be obtained with these components or a combination of them.

It is also possible to express the components of the wavefunctions using pairs of the operators $\hat{B}_+$ and $\hat{B}_-^{-1}$. In this case we can use equations (2.2), (2.3) and the relations (2.4), (2.5) to show by induction that

$$[\hat{B}_+\hat{B}_-, (\hat{B}_+\hat{B}_-^{-1})^n] = \sum_{k=1}^{2n} R(a_k)\,(\hat{B}_+\hat{B}_-^{-1})^n \tag{2.17}$$

and

$$[\hat{B}_+\hat{B}_-, (\hat{B}_-^{-1}\hat{B}_+)^n] = \sum_{k=1}^{2n} R(a_k)\,(\hat{B}_-^{-1}\hat{B}_+)^n. \tag{2.18}$$

Using these last two equations and the same conditions (2.7) and (2.11) we can show that the components of the wavefunctions, written in terms of pairs of the operators $\hat{B}_+$ and $\hat{B}_-^{-1}$, for supersymmetric shape-invariant potential barriers can be written down as

$$\Psi_-(x) = \beta\,(\hat{B}_+\hat{B}_-^{-1})^{\nu}\,\Psi_-^{(0)}(x, a_1) \tag{2.19a}$$

$$\Psi_+(x) = \gamma\,(\hat{B}_-^{-1}\hat{B}_+)^{\nu}\,\hat{B}_-^{-1}\,\Psi_+^{(0)}(x, a_0) \tag{2.19b}$$

and where ν is obtained by the relation

$$\Lambda = \sum_{k=1}^{2\nu} R(a_k). \tag{2.20}$$

Note that in a given problem either equations (2.14) or equations (2.19) could be used, but not both.

3. Asymptotic wavefunctions

The formal expressions for the components of the wavefunctions can be expressed in explicit forms if we evaluate them asymptotically. In the case of single-operator expressions we first note that using equations (1.12) and (1.13), the results (2.14) can be written as

$$\Psi_-(x) = \beta \hat{A}_+(a_1)\hat{A}_+(a_2)\cdots\hat{A}_+(a_\mu)\Psi_-^{(0)}(x, a_{\mu+1}) \tag{3.1a}$$

$$\Psi_+(x) = \gamma \hat{A}_-^{-1}(a_1)\hat{A}_-^{-1}(a_2)\cdots\hat{A}_-^{-1}(a_{\mu+1})\Psi_+^{(0)}(x, a_{\mu+1}). \tag{3.1b}$$

At this point we need to consider the two basic asymptotic behaviours for the superpotential: (a) $W(x \to \pm\infty, a_\mu)$ is constant (i.e. the potential barrier goes to a constant); and (b) $W(x \to \pm\infty, a_\mu) \longrightarrow \pm\infty$ (i.e. the potential barrier goes to $-\infty$). In the former limit the commutator

$$\left[\frac{\partial}{\partial x}, W(x, a_\mu)\right] = W'(x, a_\mu) \tag{3.2}$$

vanishes. In the latter case this commutator can be ignored as

$$\begin{aligned} W(x, a_n)\, W(x, a_k) + W'(x, a_n) &= W(x, a_n)\, W(x, a_k)\left(1 + \frac{W'(x, a_n)}{W(x, a_n)\, W(x, a_k)}\right) \\ &\to W(x, a_n)\, W(x, a_k) \end{aligned} \tag{3.3}$$

provided that $W'(x, a_n)/W(x, a_n)$ remains finite, which is the case for all realistic superpotentials. Hence in both limits we can write equations (3.1) as

$$\Psi_-(x) = \beta(W_1 + W_{\mu+1})(W_2 + W_{\mu+1})\cdots(W_\mu + W_{\mu+1})\,\Psi_-^{(0)}(x, a_{\mu+1}) \tag{3.4a}$$

$$\Psi_+(x) = \gamma(W_1 + W_{\mu+1})^{-1}(W_2 + W_{\mu+1})^{-1}\cdots(W_{\mu+1} + W_{\mu+1})^{-1}\,\Psi_+^{(0)}(x, a_{\mu+1}). \tag{3.4b}$$

In these equations the quantity W_m is the short-hand notation for $W(x, a_m)$.

If we assume that the superpotential satisfies the condition

$$W(x, a_n) = W(x, a_1) + (n-1)\,\zeta(x) \tag{3.5}$$

then these asymptotic equations can be in a form suitable for analytic continuation, that is

$$\Psi_-(x) = \beta\,\zeta^\mu\,\frac{\Gamma(2z + 2\mu)}{\Gamma(2z + \mu)}\,\Psi_-^{(0)}(x, a_{\mu+1}) \tag{3.6a}$$

$$\Psi_+(x) = \gamma\,\zeta^{-\mu-1}\,\frac{\Gamma(2z + \mu)}{\Gamma(2z + 2\mu + 1)}\,\Psi_+^{(0)}(x, a_\mu) \tag{3.6b}$$

where $z = W(x, a_1)/\zeta(x)$. The condition given by equation (3.5) is satisfied for a number of superpotentials and in the final section we give some examples. If this condition is not satisfied, the analytic continuation may still be done, but will be more complicated.

When the asymptotic behaviour of the superpotential is $W(x, a_n) \to \pm\infty$ we can use the identity

$$\lim_{y\to\pm\infty} \frac{1}{y^\mu}\frac{\Gamma(y + 2\mu)}{\Gamma(y + \mu)} = 1 \tag{3.7}$$

to express equations (3.6) in the simple form

$$\Psi_-(x) = \beta\,(2W_1)^\mu\,\Psi_-^{(0)}(x, a_{\mu+1}) \tag{3.8a}$$

$$\Psi_+(x) = \gamma\,(2W_1)^{-\mu-1}\,\Psi_+^{(0)}(x, a_{\mu+1}). \tag{3.8b}$$

We can repeat the same procedure used above in the case of a pair of operators. Again, using equations (1.12) and (1.13) the results (2.19) can be written as

$$\Psi_-(x) = \beta \prod_{k=1}^{\nu} \hat{A}_+(a_{2k-1})\, \hat{A}_-^{-1}(a_{2k})\, \Psi_-^{(0)}(x, a_{2\nu+1}) \tag{3.9a}$$

$$\Psi_+(x) = \gamma \prod_{k=1}^{\nu} \hat{A}_-^{-1}(a_{2k-1})\, \hat{A}_+(a_{2k})\, \hat{A}_-^{-1}(a_{2\nu+1})\, \Psi_+^{(0)}(x, a_{2\nu+1}). \tag{3.9b}$$

Considering the superpotential asymptotic simplifications given by equations (3.2), (3.3) and the analytic continuation condition (3.5), we can write the result for the components of the asymptotic wavefunctions in this case as

$$\Psi_-(x) = \beta \frac{\Gamma(1 - z - \nu)}{\Gamma(1 - z - 2\nu)\, \Gamma(\nu + \frac{1}{2})}\, \Psi_-^{(0)}(x, a_{2\nu+1}) \tag{3.10a}$$

$$\Psi_+(x) = \gamma \frac{\Gamma(-z - 2\nu)\, \Gamma(\nu + \frac{1}{2})}{\Gamma(1 - z - \nu)}\, \Psi_+^{(0)}(x, a_{2\nu+1}). \tag{3.10b}$$

4. General asymptotic wavefunctions and the transmission and reflection coefficients

Using the formalism developed in [17], we can write two possible asymptotic solutions for the one-dimensional time-independent Schrödinger equation in the form

$$\Psi_1(x \to \pm\infty) = A_{11}(\pm\infty)\, f_1(x) + A_{21}(\pm\infty)\, f_2(x) \tag{4.1a}$$

$$\Psi_2(x \to \pm\infty) = A_{12}(\pm\infty)\, f_1(x) + A_{22}(\pm\infty)\, f_2(x) \tag{4.1b}$$

where

$$f_1(x) = \frac{\exp(+\mathrm{i}\chi(x))}{\sqrt{q(x)}} \qquad \text{and} \qquad f_2(x) = \frac{\exp(-\mathrm{i}\chi(x))}{\sqrt{q(x)}} \tag{4.2}$$

with

$$\chi(x) = \int^x q(\xi)\, \mathrm{d}\xi \qquad \text{and} \qquad q(x) = \frac{\sqrt{2m}}{\hbar}\, W(x, a_{\mu+1}). \tag{4.3}$$

If we define the vectors

$$|\Psi(x)\rangle = \begin{bmatrix} |\Psi_1(x)\rangle \\ |\Psi_2(x)\rangle \end{bmatrix} \tag{4.4}$$

and

$$\langle f(x)| = [\, \langle f_1(x)| \;\; \langle f_2(x)|\,] \tag{4.5}$$

then we can write

$$\langle \Psi(x \to \pm\infty)| = \langle f(x)| \boldsymbol{A}(\pm\infty) \tag{4.6}$$

where the asymptotic coefficients matrix is given by

$$\boldsymbol{A}(\pm\infty) = \begin{bmatrix} A_{11}(\pm\infty) & A_{12}(\pm\infty) \\ A_{21}(\pm\infty) & A_{22}(\pm\infty) \end{bmatrix}. \tag{4.7}$$

Considering that the space evolution of the coefficients matrix $\boldsymbol{A}$, given by an iteration process, can be written as

$$\boldsymbol{A}(x) = \boldsymbol{F}(x, x_0)\, \boldsymbol{A}(x_0) \tag{4.8}$$

then if we know the asymptotic coefficients in $-\infty$ and $+\infty$, we can obtain the evolution matrix

$$\boldsymbol{F}(-\infty, +\infty) = \boldsymbol{A}(-\infty)\,\boldsymbol{A}^{-1}(+\infty). \tag{4.9}$$

Using equation (4.9) we can show two basic properties of the $\boldsymbol{F}$-matrix:

$$\det\ \boldsymbol{F}(-\infty, +\infty) = 1 \tag{4.10a}$$

$$\boldsymbol{F}(-\infty, +\infty) = \boldsymbol{F}^{-1}(+\infty, -\infty). \tag{4.10b}$$

The knowledge of $\boldsymbol{F}(-\infty, +\infty)$ permits the determination of the exact transmission and reflection barrier coefficients. If we consider a wave incidence from $-\infty$ to $+\infty$ then we can write the asymptotic wavefunction in the form

$$\Psi(x \to -\infty) = f_1(x) + C_R\, f_2(x) \tag{4.11a}$$

$$\Psi(x \to +\infty) = C_T\, f_2(x) \tag{4.11b}$$

or

$$\Psi(x \to -\infty) = \langle f(x)|a(-\infty)\rangle \tag{4.12a}$$

$$\Psi(x \to +\infty) = \langle f(x)|a(+\infty)\rangle \tag{4.12b}$$

where

$$|a(-\infty)\rangle = \begin{bmatrix} 1 \\ C_R \end{bmatrix} \qquad \text{and} \qquad |a(+\infty)\rangle = \begin{bmatrix} 0 \\ C_T \end{bmatrix}. \tag{4.13}$$

Considering that

$$|a(-\infty)\rangle = \boldsymbol{F}(-\infty, +\infty)\,|a(+\infty)\rangle \tag{4.14}$$

we can conclude that

$$C_T = \frac{1}{F_{12}(-\infty, +\infty)} \tag{4.15a}$$

$$C_R = \frac{F_{22}(-\infty, +\infty)}{F_{12}(-\infty, +\infty)} \tag{4.15b}$$

and the transmission and reflection coefficients are given by $T = |C_T|^2$ and $R = |C_R|^2$. At this point if we consider the conservation of probability, the time-reversal invariance and the invariance under space reflection, it is possible to show the additional $\boldsymbol{F}$-matrix properties:

$$F_{11}(-\infty, +\infty) = -F_{22}(-\infty, +\infty) \tag{4.16a}$$

$$F_{21}(-\infty, +\infty) = -F_{12}(-\infty, +\infty) \tag{4.16b}$$

$$|F_{12}(-\infty, +\infty)| = |F_{12}(+\infty, -\infty)| \geqslant 1 \tag{4.16c}$$

$$|F_{22}(-\infty, +\infty)| = |F_{22}(+\infty, -\infty)| \leqslant |F_{12}(-\infty, +\infty)|. \tag{4.16d}$$

5. Applications

5.1. Parabolic barrier

For a parabolic potential barrier [18]

$$V_1(x) = V_0 - \tfrac{1}{2} m\Omega^2 x^2 \tag{5.1}$$

the corresponding superpotential, obtained by using equation (1.7), is given by

$$W(x, a_1) = a_1 x \tag{5.2}$$

where $a_1 = \pm \mathrm{i}\sqrt{m/2}\,\Omega$. The shape-invariance condition (1.9) implies that

$$R(a_n) = \pm 2\mathrm{i}\varepsilon_0 \tag{5.3}$$

where $\varepsilon_0 = \frac{1}{2}\hbar\Omega$. Using equation (2.20) we can conclude that

$$\sum_{k=1}^{\mu} R(a_k) = \pm \mathrm{i}2\mu\varepsilon_0 = \Lambda = E - V_0 \mp \mathrm{i}\varepsilon_0 \tag{5.4}$$

or

$$\mu = -\tfrac{1}{2} \pm \mathrm{i}\tfrac{1}{2}\lambda \tag{5.5}$$

where $\lambda = (V_0 - E)/\varepsilon_0$. The asymptotic form for the components of the wavefunction can be obtained using these results in equations (3.8),

$$\Psi_-(x) = \beta\,(\pm \mathrm{i})^{-\frac{1}{2}\pm \mathrm{i}\lambda/2}\,\varrho^{\pm \mathrm{i}\lambda/2}\left[\frac{\exp(\mp \mathrm{i}\varrho^2)}{\sqrt{\varrho}}\right] \tag{5.6a}$$

$$\Psi_+(x) = \gamma\,(\pm \mathrm{i})^{-\frac{1}{2}\mp \mathrm{i}\lambda/2}\,\varrho^{\mp \mathrm{i}\lambda/2}\left[\frac{\exp(\pm \mathrm{i}\varrho^2)}{\sqrt{\varrho}}\right] \tag{5.6b}$$

or

$$\Psi_-(x) = \beta \left\{ \begin{array}{c} \mathrm{e}^{\mp \mathrm{i}(k+\frac{1}{4})\pi}\,\mathrm{e}^{-(k+\frac{1}{4})\pi\lambda} \\ \mathrm{e}^{\pm \mathrm{i}(k+\frac{3}{4})\pi}\,\mathrm{e}^{(k+\frac{3}{4})\pi\lambda} \end{array} \right\} \varrho^{\pm \mathrm{i}\lambda/2}\left[\frac{\exp(\mp \mathrm{i}\varrho^2)}{\sqrt{\varrho}}\right] \tag{5.7}$$

and

$$\Psi_+(x) = \gamma \left\{ \begin{array}{c} \mathrm{e}^{\mp \mathrm{i}(k+\frac{1}{4})\pi}\,\mathrm{e}^{(k+\frac{1}{4})\pi\lambda} \\ \mathrm{e}^{\pm \mathrm{i}(k+\frac{3}{4})\pi}\,\mathrm{e}^{-(k+\frac{3}{4})\pi\lambda} \end{array} \right\} \varrho^{\mp \mathrm{i}\lambda/2}\left[\frac{\exp(\pm \mathrm{i}\varrho^2)}{\sqrt{\varrho}}\right] \tag{5.8}$$

where $\varrho = \sqrt{m\Omega/2\hbar}\,x$ and $k = 0, 1, 2, \ldots$. Using these results it is possible to write two asymptotic solutions of the Schrödinger equation when $x \to +\infty$ in the form

$$\Psi_1(x \to +\infty) = \mathrm{e}^{\mathrm{i}\pi/4}\mathrm{e}^{-\pi\lambda/4}|\varrho|^{\mathrm{i}\lambda/2}\left[\frac{\exp(-\mathrm{i}\varrho^2)}{\sqrt{\varrho}}\right] + \mathrm{e}^{-\mathrm{i}\pi/4}\mathrm{e}^{-\pi\lambda/4}|\varrho|^{-\mathrm{i}\lambda/2}\left[\frac{\exp(+\mathrm{i}\varrho^2)}{\sqrt{\varrho}}\right] \tag{5.9a}$$

$$\Psi_2(x \to +\infty) = \mathrm{e}^{-\mathrm{i}\pi/4}\mathrm{e}^{-3\pi\lambda/4}|\varrho|^{\mathrm{i}\lambda/2}\left[\frac{\exp(-\mathrm{i}\varrho^2)}{\sqrt{\varrho}}\right] + \mathrm{e}^{\mathrm{i}\pi/4}\mathrm{e}^{-3\pi\lambda/4}|\varrho|^{-\mathrm{i}\lambda/2}\left[\frac{\exp(+\mathrm{i}\varrho^2)}{\sqrt{\varrho}}\right] \tag{5.9b}$$

therefore, if we identify

$$f_1(x) = \frac{\exp(-\mathrm{i}\varrho^2)}{\sqrt{\varrho}} \qquad \text{and} \qquad f_2(x) = \frac{\exp(+\mathrm{i}\varrho^2)}{\sqrt{\varrho}} \tag{5.10}$$

we can conclude that the elements of the $\boldsymbol{A}(+\infty)$-matrix will be

$$A_{11}(+\infty) = \mathrm{e}^{\mathrm{i}\pi/4}\mathrm{e}^{-\pi\lambda/4}|\varrho|^{\mathrm{i}\lambda/2} \tag{5.11a}$$

$$A_{12}(+\infty) = \mathrm{e}^{-\mathrm{i}\pi/4}\mathrm{e}^{-3\pi\lambda/4}|\varrho|^{\mathrm{i}\lambda/2} \tag{5.11b}$$

$$A_{21}(+\infty) = \mathrm{e}^{-\mathrm{i}\pi/4}\mathrm{e}^{-\pi\lambda/4}|\varrho|^{-\mathrm{i}\lambda/2} \tag{5.11c}$$

$$A_{22}(+\infty) = \mathrm{e}^{\mathrm{i}\pi/4}\mathrm{e}^{-3\pi\lambda/4}|\varrho|^{-\mathrm{i}\lambda/2}. \tag{5.11d}$$

In the case of $x \to -\infty$ if we consider that

$$\varrho^{\pm i\lambda/2} = e^{\mp(n+\frac{1}{2})\pi\lambda}|\varrho|^{\pm i\lambda/2} \qquad n = 0, 1, 2, \ldots \tag{5.12}$$

in equations (5.8), then we can write two asymptotic solutions of the Schrödinger equation when $x \to -\infty$ in the form

$$\Psi_1(x \to -\infty) = \left[(e^{i3\pi/4} + e^{i\pi/4})\, e^{\pi\lambda/4} + e^{-i\pi/4}e^{-\pi\lambda/4}\right]|\varrho|^{i\lambda/2}\left[\frac{\exp(-i\varrho^2)}{\sqrt{\varrho}}\right]$$
$$+\left[(e^{i3\pi/4} + e^{i\pi/4})\, e^{\pi\lambda/4} - e^{i\pi/4}e^{-\pi\lambda/4}\right]|\varrho|^{-i\lambda/2}\left[\frac{\exp(+i\varrho^2)}{\sqrt{\varrho}}\right] \tag{5.13}$$

and

$$\Psi_2(x \to -\infty) = \left[(e^{i3\pi/4} + e^{i\pi/4})\, e^{-\pi\lambda/4} + e^{i\pi/4}e^{-3\pi\lambda/4}\right]|\varrho|^{i\lambda/2}\left[\frac{\exp(-i\varrho^2)}{\sqrt{\varrho}}\right]$$
$$+\left[(e^{i3\pi/4} + e^{i\pi/4})\, e^{-\pi\lambda/4} - e^{-i\pi/4}e^{-3\pi\lambda/4}\right]|\varrho|^{-i\lambda/2}\left[\frac{\exp(+i\varrho^2)}{\sqrt{\varrho}}\right] \tag{5.14}$$

therefore we can identify the elements of the $\boldsymbol{A}(-\infty)$-matrix as

$$A_{11}(-\infty) = \left[(e^{i3\pi/4} + e^{i\pi/4})\, e^{\pi\lambda/4} + e^{-i\pi/4}e^{-\pi\lambda/4}\right]|\varrho|^{i\lambda/2} \tag{5.15a}$$

$$A_{12}(-\infty) = \left[(e^{i3\pi/4} + e^{i\pi/4})\, e^{-\pi\lambda/4} + e^{i\pi/4}e^{-3\pi\lambda/4}\right]|\varrho|^{i\lambda/2} \tag{5.15b}$$

$$A_{21}(-\infty) = \left[(e^{i3\pi/4} + e^{i\pi/4})\, e^{\pi\lambda/4} - e^{i\pi/4}e^{-\pi\lambda/4}\right]|\varrho|^{-i\lambda/2} \tag{5.15c}$$

$$A_{22}(-\infty) = \left[(e^{i3\pi/4} + e^{i\pi/4})\, e^{-\pi\lambda/4} - e^{-i\pi/4}e^{-3\pi\lambda/4}\right]|\varrho|^{-i\lambda/2}. \tag{5.15d}$$

In the choice of the two asymptotic wavefunctions for $x \to \pm\infty$ we considered the set of properties given by equations (4.10) and (4.16) that the $\boldsymbol{F}$-matrix needs to satisfy. Using the results for $\boldsymbol{A}(-\infty)$ and $\boldsymbol{A}(+\infty)$ in equations (4.9), we can show that the evolution matrix can be written as

$$\boldsymbol{F}(-\infty, +\infty) = \begin{bmatrix} ie^{\pi\lambda/2} & (1 + ie^{\pi\lambda/2})|\varrho|^{i\lambda} \\ (-1 + ie^{\pi\lambda/2})|\varrho|^{-i\lambda} & ie^{\pi\lambda/2} \end{bmatrix} \tag{5.16}$$

and the exact transmission and reflection coefficients are given by

$$T = \frac{1}{|F_{12}(-\infty, +\infty)|^2} = \frac{1}{1 + e^{\pi\lambda}} \tag{5.17}$$

and

$$R = \frac{|F_{22}(-\infty, +\infty)|^2}{|F_{12}(-\infty, +\infty)|^2} = \frac{e^{\pi\lambda}}{1 + e^{\pi\lambda}}. \tag{5.18}$$

5.2. Morse barrier

For a Morse potential barrier [19]

$$V_1(x) = V_0\left(2e^{x/b} - e^{2x/b}\right) \tag{5.19}$$

the corresponding superpotential, obtained by equation (1.7), is given by

$$W(x, a_1) = a_1 + \alpha\, e^{x/b} \tag{5.20}$$

where

$$a_1 = \sqrt{\varepsilon}(1 \mp \mathrm{i}s)$$
$$\alpha = \pm\mathrm{i}\sqrt{V_0} \tag{5.21}$$

with $\varepsilon = \hbar^2/(8mb^2)$ and $s = \sqrt{V_0/\varepsilon}$. The shape-invariance condition (1.9) implies that

$$R(a_n) = a_n^2 - a_{n+1}^2 \tag{5.22}$$

where $a_{n+1} = a_n + 2\sqrt{\varepsilon}$. Using equation (2.20) we can conclude that

$$\sum_{k=1}^{\mu} R(a_k) = a_1^2 - a_{\mu+1}^2 = \Lambda = E + a_1^2 \tag{5.23}$$

or

$$a_{\mu+1} = \pm\mathrm{i}\sqrt{E}. \tag{5.24}$$

If we remember that

$$\mu = \frac{a_{\mu+1} - a_1}{2\sqrt{\varepsilon}} \tag{5.25}$$

we can use equations (5.21) and (5.24) to show that

$$\mu = -\tfrac{1}{2} \pm \mathrm{i}\tfrac{1}{2}s \pm \mathrm{i}\tfrac{1}{2}r \tag{5.26}$$

where $r = \sqrt{E/\varepsilon}$.

Considering the asymmetry of the Morse potential barrier, the wavefunction will have a different behaviour in $+\infty$ and $-\infty$. Therefore, the asymptotic form of the components of the wavefunction for $x \to +\infty$ can be obtained using the last results in equations (3.8),

$$\Psi_-(x \to +\infty) = \beta\, \mathrm{e}^{\mp\mathrm{i}\pi/4}\, \mathrm{e}^{-\pi(s\pm r)/4} \left[\frac{\exp\left(\mp\mathrm{i}\frac{1}{2}s\exp(x/b)\right)}{\sqrt{\exp(x/b)}}\right] \tag{5.27a}$$

$$\Psi_+(x \to +\infty) = \gamma\, \mathrm{e}^{\mp\mathrm{i}\pi/4}\, \mathrm{e}^{\pi(s\pm r)/4} \left[\frac{\exp\left(\pm\mathrm{i}\frac{1}{2}s\exp(x/b)\right)}{\sqrt{\exp(x/b)}}\right]. \tag{5.27b}$$

Using these results it is possible to write two asymptotic solutions of the Schrödinger equation when $x \to +\infty$ in the form

$$\Psi_1(x \to +\infty) = \mathrm{e}^{\mathrm{i}\pi/4}\mathrm{e}^{-\pi(s+r)/4}\left[\frac{\exp\left(-\mathrm{i}\frac{1}{2}s\exp(x/b)\right)}{\sqrt{\exp(x/b)}}\right] + \mathrm{e}^{-\mathrm{i}\pi/4}\mathrm{e}^{\pi(s-r)/4}\left[\frac{\exp\left(+\mathrm{i}\frac{1}{2}s\exp(x/b)\right)}{\sqrt{\exp(x/b)}}\right] \tag{5.28a}$$

$$\Psi_2(x \to +\infty) = \mathrm{e}^{-\mathrm{i}\pi/4}\mathrm{e}^{-\pi(s-r)/4}\left[\frac{\exp\left(-\mathrm{i}\frac{1}{2}s\exp(x/b)\right)}{\sqrt{\exp(x/b)}}\right] + \mathrm{e}^{\mathrm{i}\pi/4}\mathrm{e}^{\pi(s+r)/4}\left[\frac{\exp\left(+\mathrm{i}\frac{1}{2}s\exp(x/b)\right)}{\sqrt{\exp(x/b)}}\right] \tag{5.28b}$$

therefore, if we identify

$$f_1(x \to +\infty) = \frac{\exp\left(-\mathrm{i}\frac{1}{2}s\exp(x/b)\right)}{\sqrt{\exp(x/b)}}$$
$$f_2(x \to +\infty) = \frac{\exp\left(+\mathrm{i}\frac{1}{2}s\exp(x/b)\right)}{\sqrt{\exp(x/b)}} \tag{5.29}$$

we can conclude that the elements of the $\boldsymbol{A}(+\infty)$-matrix will be

$$A_{11}(+\infty) = \mathrm{e}^{\mathrm{i}\pi/4}\mathrm{e}^{-\pi(s+r)/4} \tag{5.30a}$$

$$A_{12}(+\infty) = \mathrm{e}^{-\mathrm{i}\pi/4}\mathrm{e}^{-\pi(s-r)/4} \tag{5.30b}$$

$$A_{21}(+\infty) = \mathrm{e}^{-\mathrm{i}\pi/4}\mathrm{e}^{\pi(s-r)/4} \tag{5.30c}$$

$$A_{22}(+\infty) = \mathrm{e}^{\mathrm{i}\pi/4}\mathrm{e}^{\pi(s+r)/4}. \tag{5.30d}$$

In the case of $x \to -\infty$ we can substitute for s and r and using equation (3.6) find

$$\Psi_-(x \to -\infty) = \beta \frac{\sqrt{r}\,\Gamma(\pm \mathrm{i}r)}{\Gamma\left(\frac{1}{2} \pm \mathrm{i}\frac{1}{2}s \pm \mathrm{i}\frac{1}{2}r\right)} \mathrm{e}^{\mp \mathrm{i}kx} \tag{5.31a}$$

$$\Psi_+(x \to -\infty) = \gamma \frac{\Gamma\left(\frac{1}{2} \pm \mathrm{i}\frac{1}{2}s \pm \mathrm{i}\frac{1}{2}r\right)}{\sqrt{r}\,\Gamma(\pm \mathrm{i}r)} \mathrm{e}^{\pm \mathrm{i}kx} \tag{5.31b}$$

where $k = \sqrt{2mE}/\hbar$. Using these results it is possible to write two asymptotic solutions of the Schrödinger equation when $x \to -\infty$ in the form

$$\begin{aligned}\Psi_1(x \to -\infty) = {} & \left[\frac{\mathrm{e}^{\mathrm{i}3\pi/4}\,\mathrm{e}^{-\pi r/2}}{\Gamma\left(\frac{1}{2} + \mathrm{i}\frac{1}{2}s + \mathrm{i}\frac{1}{2}r\right)} + \frac{\mathrm{e}^{\mathrm{i}\pi/4}\,\mathrm{e}^{\pi s/2}}{\Gamma\left(\frac{1}{2} + \mathrm{i}\frac{1}{2}s - \mathrm{i}\frac{1}{2}r\right)}\right]\sqrt{r}\,\Gamma(\mathrm{i}r)\,\mathrm{e}^{\mathrm{i}kx} \\ & + \left[\frac{\mathrm{e}^{\mathrm{i}\pi/4}\,\mathrm{e}^{\pi(s-r)/2}}{\Gamma\left(\frac{1}{2} - \mathrm{i}\frac{1}{2}s - \mathrm{i}\frac{1}{2}r\right)} + \frac{\mathrm{e}^{\mathrm{i}3\pi/4}}{\Gamma\left(\frac{1}{2} - \mathrm{i}\frac{1}{2}s + \mathrm{i}\frac{1}{2}r\right)}\right]\sqrt{r}\,\Gamma(\mathrm{i}r)\,\mathrm{e}^{-\mathrm{i}kx}\end{aligned} \tag{5.32}$$

and

$$\begin{aligned}\Psi_2(x \to -\infty) = {} & \left[\frac{\mathrm{e}^{\mathrm{i}\pi/4}}{\Gamma\left(\frac{1}{2} + \mathrm{i}\frac{1}{2}s + \mathrm{i}\frac{1}{2}r\right)} + \frac{\mathrm{e}^{\mathrm{i}3\pi/4}\,\mathrm{e}^{\pi(s+r)/2}}{\Gamma\left(\frac{1}{2} + \mathrm{i}\frac{1}{2}s - \mathrm{i}\frac{1}{2}r\right)}\right]\sqrt{r}\,\Gamma(-\mathrm{i}r)\,\mathrm{e}^{\mathrm{i}kx} \\ & + \left[\frac{\mathrm{e}^{\mathrm{i}\pi/4}\,\mathrm{e}^{\pi r/2}}{\Gamma\left(\frac{1}{2} - \mathrm{i}\frac{1}{2}s - \mathrm{i}\frac{1}{2}r\right)} + \frac{\mathrm{e}^{\mathrm{i}3\pi/4}\,\mathrm{e}^{\pi s/2}}{\Gamma\left(\frac{1}{2} - \mathrm{i}\frac{1}{2}s - \mathrm{i}\frac{1}{2}r\right)}\right]\sqrt{r}\,\Gamma(-\mathrm{i}r)\,\mathrm{e}^{-\mathrm{i}kx}\end{aligned} \tag{5.33}$$

therefore, if we identify

$$f_1(x \to -\infty) = \frac{\exp(+\mathrm{i}kx)}{\sqrt{k}} \qquad \text{and} \qquad f_2(x \to -\infty) = \frac{\exp(-\mathrm{i}kx)}{\sqrt{k}} \tag{5.34}$$

we can conclude that the elements of the $\boldsymbol{A}(-\infty)$-matrix will be

$$A_{11}(-\infty) = \left[\frac{\mathrm{e}^{\mathrm{i}3\pi/4}\,\mathrm{e}^{-\pi r/2}}{\Gamma\left(\frac{1}{2} + \mathrm{i}\frac{1}{2}s + \mathrm{i}\frac{1}{2}r\right)} + \frac{\mathrm{e}^{\mathrm{i}\pi/4}\,\mathrm{e}^{\pi s/2}}{\Gamma\left(\frac{1}{2} + \mathrm{i}\frac{1}{2}s - \mathrm{i}\frac{1}{2}r\right)}\right]\sqrt{r}\,\Gamma(\mathrm{i}r) \tag{5.35a}$$

$$A_{12}(-\infty) = \left[\frac{\mathrm{e}^{\mathrm{i}\pi/4}}{\Gamma\left(\frac{1}{2} + \mathrm{i}\frac{1}{2}s + \mathrm{i}\frac{1}{2}r\right)} + \frac{\mathrm{e}^{\mathrm{i}3\pi/4}\,\mathrm{e}^{\pi(s+r)/2}}{\Gamma\left(\frac{1}{2} + \mathrm{i}\frac{1}{2}s - \mathrm{i}\frac{1}{2}r\right)}\right]\sqrt{r}\,\Gamma(-\mathrm{i}r) \tag{5.35b}$$

$$A_{21}(-\infty) = \left[\frac{\mathrm{e}^{\mathrm{i}\pi/4}\,\mathrm{e}^{\pi(s-r)/2}}{\Gamma\left(\frac{1}{2} - \mathrm{i}\frac{1}{2}s - \mathrm{i}\frac{1}{2}r\right)} + \frac{\mathrm{e}^{\mathrm{i}3\pi/4}}{\Gamma\left(\frac{1}{2} - \mathrm{i}\frac{1}{2}s + \mathrm{i}\frac{1}{2}r\right)}\right]\sqrt{r}\,\Gamma(\mathrm{i}r) \tag{5.35c}$$

$$A_{22}(-\infty) = \left[\frac{\mathrm{e}^{\mathrm{i}\pi/4}\,\mathrm{e}^{\pi r/2}}{\Gamma\left(\frac{1}{2} - \mathrm{i}\frac{1}{2}s - \mathrm{i}\frac{1}{2}r\right)} + \frac{\mathrm{e}^{\mathrm{i}3\pi/4}\,\mathrm{e}^{\pi s/2}}{\Gamma\left(\frac{1}{2} - \mathrm{i}\frac{1}{2}s - \mathrm{i}\frac{1}{2}r\right)}\right]\sqrt{r}\,\Gamma(-\mathrm{i}r). \tag{5.35d}$$

Again, in the choice of the two asymptotic wavefunctions for $x \to \pm\infty$ we have considered the properties of the $\boldsymbol{F}$-matrix. Using the results for $\boldsymbol{A}(-\infty)$ and $\boldsymbol{A}(+\infty)$ in equations (4.9) we can show that the evolution matrix can be written as

$$\boldsymbol{F}(-\infty, +\infty) = \begin{bmatrix} \mathrm{i}g & \mathrm{i}h \\ \mathrm{i}h^* & \mathrm{i}g^* \end{bmatrix} \tag{5.36}$$

where

$$g = \frac{\mathrm{e}^{\pi(s-r)/4}\sqrt{r}\,\Gamma(\mathrm{i}r)}{\Gamma\left(\frac{1}{2} + \mathrm{i}\frac{1}{2}s + \mathrm{i}\frac{1}{2}r\right)} \tag{5.37}$$

and

$$h = \frac{\mathrm{e}^{\pi(s+r)/4}\sqrt{r}\,\Gamma(\mathrm{i}r)}{\Gamma\left(\frac{1}{2} + \mathrm{i}\frac{1}{2}s - \mathrm{i}\frac{1}{2}r\right)}. \tag{5.38}$$

In this case the exact transmission and reflection coefficients are given by

$$T = \frac{1}{|F_{12}(-\infty, +\infty)|^2} = \frac{\mathrm{e}^{-\pi(s+r)/2}\sinh(\pi r)}{\cosh\left[\frac{1}{2}\pi(s-r)\right]} \tag{5.39}$$

and

$$R = \frac{|F_{22}(-\infty, +\infty)|^2}{|F_{12}(-\infty, +\infty)|^2} = \frac{\mathrm{e}^{-\pi r}\cosh[\pi/2(s+r)]}{\cosh[\pi/2(s-r)]}. \tag{5.40}$$

5.3. Eckart barrier

For an Eckart potential barrier [20]

$$V_1(x) = V_0\,\mathrm{sech}^2(x/2b) \tag{5.41}$$

the corresponding superpotential, obtained from equation (1.7), is given by

$$W(x, a_1) = a_1 \tanh(x/2b) \tag{5.42}$$

where

$$a_1 = \sqrt{\varepsilon}\,(-1 \pm \mathrm{i}s) \tag{5.43}$$

with $\varepsilon = \hbar^2/(32mb^2)$ and $s = \sqrt{V_0/\varepsilon - 1}$. The shape-invariance condition (1.9) implies that

$$R(a_n) = a_n^2 - a_{n+1}^2 \tag{5.44}$$

where $a_{n+1} = a_n - 2\sqrt{\varepsilon}$. Using equation (2.20) we can conclude that

$$\sum_{k=1}^{2\nu} R(a_k) = a_1^2 - a_{2\nu+1}^2 = \Lambda = E + a_1^2 \tag{5.45}$$

or

$$a_{2\nu+1} = \pm\mathrm{i}\sqrt{E}. \tag{5.46}$$

If we remember that

$$2\nu = \frac{a_{2\nu+1} - a_1}{-2\sqrt{\varepsilon}} \tag{5.47}$$

we can use equations (5.43) and (5.46) to show that

$$2\nu = -\tfrac{1}{2} \pm \mathrm{i}\tfrac{1}{2}s \mp \mathrm{i}\tfrac{1}{2}r \tag{5.48}$$

where $r = \sqrt{E/\varepsilon}$.

The asymptotic form of the components of the wavefunction can be obtained using these results in equations (3.10),

$$\Psi_-(x) = \beta \frac{\Gamma\left(\frac{3}{4} \pm \mathrm{i}\frac{1}{4}s \pm \mathrm{i}\frac{1}{4}r\right)}{\Gamma\left(1 \pm \mathrm{i}\frac{1}{2}r\right)\Gamma\left(\frac{1}{4} \pm \mathrm{i}\frac{1}{4}s \mp \mathrm{i}\frac{1}{4}r\right)} \mathrm{e}^{\mp \mathrm{i}kx} \tag{5.49a}$$

$$\Psi_+(x) = \gamma \frac{\Gamma\left(\pm\mathrm{i}\frac{1}{2}r\right)\Gamma\left(\frac{1}{4} \pm \mathrm{i}\frac{1}{4}s \mp \mathrm{i}\frac{1}{4}r\right)}{\Gamma\left(\frac{3}{4} \pm \mathrm{i}\frac{1}{4}s \pm \mathrm{i}\frac{1}{4}r\right)} \mathrm{e}^{\pm \mathrm{i}kx} \tag{5.49b}$$

where $k = \sqrt{2mE}/\hbar$. Using these results and the relation

$$\Gamma\left(\tfrac{1}{4} \pm \mathrm{i}y\right)\Gamma\left(\tfrac{3}{4} \mp \mathrm{i}y\right) = \frac{\sqrt{2}\pi}{\cosh(\pi y) \pm \sinh(\pi y)} \tag{5.50}$$

it is possible to write two asymptotic solutions of the Schrödinger equation when $x \to +\infty$ in the form

$$\Psi_1(x \to +\infty) = C_1 \mathrm{e}^{-\mathrm{i}kx} + C_1^* \mathrm{e}^{\mathrm{i}kx} \tag{5.51a}$$

$$\Psi_2(x \to +\infty) = C_2^* \mathrm{e}^{-\mathrm{i}kx} - C_2 \mathrm{e}^{\mathrm{i}kx} \tag{5.51b}$$

where

$$C_1 = \sqrt{2}\,\pi \left\{\cosh\left[\tfrac{1}{4}\pi(r \pm s)\right] + \mathrm{i}\sinh\left[\tfrac{1}{4}\pi(r \pm s)\right]\right\}^{-1} \times \left\{\Gamma\left(1 + \mathrm{i}\tfrac{1}{2}r\right)\Gamma\left(\tfrac{1}{4} \pm \mathrm{i}\tfrac{1}{4}s - \mathrm{i}\tfrac{1}{4}r\right)\Gamma\left(\tfrac{1}{4} \mp \mathrm{i}\tfrac{1}{4}s - \mathrm{i}\tfrac{1}{4}r\right)\right\}^{-1} \tag{5.52}$$

and

$$C_2 = \sqrt{2}\,\pi\,\Gamma\left(\mathrm{i}\tfrac{1}{2}r\right)\left\{\cosh\left[\tfrac{1}{4}\pi(r \pm s)\right] - \mathrm{i}\sinh\left[\tfrac{1}{4}\pi(r \pm s)\right]\right\}^{-1} \times \left\{\Gamma\left(\tfrac{3}{4} \mp \mathrm{i}\tfrac{1}{4}s + \mathrm{i}\tfrac{1}{4}r\right)\Gamma\left(\tfrac{3}{4} \pm \mathrm{i}\tfrac{1}{4}s + \mathrm{i}\tfrac{1}{4}r\right)\right\}^{-1}. \tag{5.53}$$

Therefore, if we identify in equation (5.51)

$$f_1(x \to +\infty) = \frac{\mathrm{e}^{-\mathrm{i}kx}}{\sqrt{k}} \qquad \text{and} \qquad f_2(x \to +\infty) = \frac{\mathrm{e}^{+\mathrm{i}kx}}{\sqrt{k}} \tag{5.54}$$

we can conclude that the elements of the $\boldsymbol{A}(+\infty)$-matrix will be

$$A_{11}(+\infty) = C_1 \tag{5.55a}$$

$$A_{12}(+\infty) = C_2^* \tag{5.55b}$$

$$A_{21}(+\infty) = C_1^* \tag{5.55c}$$

$$A_{22}(+\infty) = -C_2. \tag{5.55d}$$

Also we can write two asymptotic solutions of the Schrödinger equation when $x \to -\infty$ in the form

$$\Psi_1(x \to -\infty) = C_1 \mathrm{e}^{\mathrm{i}kx} - C_1^* \mathrm{e}^{-\mathrm{i}kx} \tag{5.56a}$$

$$\Psi_2(x \to -\infty) = -C_2^* \mathrm{e}^{\mathrm{i}kx} - C_2 \mathrm{e}^{-\mathrm{i}kx} \tag{5.56b}$$

and identifying

$$f_1(x \to -\infty) = \frac{\mathrm{e}^{+\mathrm{i}kx}}{\sqrt{k}} \qquad \text{and} \qquad f_2(x \to -\infty) = \frac{\mathrm{e}^{-\mathrm{i}kx}}{\sqrt{k}} \tag{5.57}$$

we can conclude that the elements of the $\boldsymbol{A}(-\infty)$-matrix will be

$$A_{11}(-\infty) = C_1 \tag{5.58a}$$

$$A_{12}(-\infty) = -C_2^* \tag{5.58b}$$

$$A_{21}(-\infty) = -C_1^* \tag{5.58c}$$

$$A_{22}(-\infty) = -C_2. \tag{5.58d}$$

Using the results for $\boldsymbol{A}(-\infty)$ and $\boldsymbol{A}(+\infty)$ in equation (4.9) we can show that the evolution matrix can be written as

$$\boldsymbol{F}(-\infty, +\infty) = \begin{bmatrix} g & h \\ -h^* & -g^* \end{bmatrix} \tag{5.59}$$

where

$$g = \frac{C_1C_2 - C_1^*C_2^*}{C_1C_2 + C_1^*C_2^*} \tag{5.60}$$

and

$$h = \frac{2C_1C_2^*}{C_1C_2 + C_1^*C_2^*}. \tag{5.61}$$

On using equations (5.52) and (5.53), after a considerable amount of algebra we can show that the exact transmission and reflection coefficients are given by

$$T = \frac{1}{|F_{12}(-\infty, +\infty)|^2} = \frac{1}{4}\left|\frac{C_2}{C_2^*} + \frac{C_1^*}{C_1}\right|^2 = \frac{\sinh^2(\pi r/2)}{\sinh^2(\pi r/2) + \cosh^2(\pi s/2)} \tag{5.62}$$

and

$$R = \frac{|F_{22}(-\infty, +\infty)|^2}{|F_{12}(-\infty, +\infty)|^2} = \frac{1}{4}\left|\frac{C_2}{C_2^*} - \frac{C_1^*}{C_1}\right|^2 = \frac{\cosh^2(\pi s/2)}{\sinh^2(\pi r/2) + \cosh^2(\pi s/2)}. \tag{5.63}$$

Here we should note that a special form of the Eckart potential

$$V(x) = -n(n+1)\mathrm{sech}^2(x/2b) \tag{5.64}$$

where n is an integer, is reflectionless as the Hamiltonian with this potential is the supersymmetric partner of a free particle [3]. The well studied Korteweg–de Vries equation possesses these potentials and other reflectionless multisoliton solutions [21]. It is straightforward to show that our result in equation (5.63) indeed vanishes when $V_0 = -n(n+1)$.

6. Concluding remarks

In conclusion, we note that the present technique is a powerful and an elegant prescription to obtain exact reflection and transmission coefficients. This method may also be used for all supersymmetric shape-invariant potential barriers that satisfy the analytic continuation condition (3.5).

One possible application of the shape-invariance formalism is to multidimensional quantum tunnelling. In nuclear physics applications multidimensional quantum tunnelling can be visualized as the tunnelling of a quantum mechanical system (such as a nucleus with internal excitation) instead of a structureless particle through a one-dimensional barrier. The nucleus is typically taken to enter the barrier in its ground state and may emerge either in the ground state or in an excited state on the other side of the barrier. The interaction between the penetrating quantum system and the barrier also needs to be specified based on the physical conditions of the problem. It has been known for some time that, if the excitation energies are neglected, the penetration probability of an N-dimensional system can be reduced to a sum of probabilities of N one-dimensional suitably defined barriers [22]. The eigenchannel formulation remains valid even for finite excitation energies as long as the energy dependence of the weight factors is taken into account [23]. The eigenchannel approach was shown to be appropriate for the description of fusion reactions of deformed nuclei below the Coulomb barrier [1]. Our formulation would be applicable in such cases if the eigenpotentials are shape invariant. A simpler limit would assume factorization of the interaction between the barrier and the quantum system into a product of two quantities which are functions of the barrier and internal degrees of freedom, respectively. Such a factorization approach was already applied to a coupled system of equations for bound states [24].

Our approach, in general, could be applied to other continuum problems besides tunnelling. For example, the Coulomb problem was shown to be shape invariant and consequently Coulomb scattering can be treated using the methods discussed here in addition to the standard approach using supersymmetric quantum mechanics [25].

Our formulation also casts the tunnelling problem in an algebraic basis [5, 26]. If the internal system can be described by an algebraic model such as the interacting boson model [27] then it may be possible to cast the entire problem into an algebraic framework. A group-theoretical formulation can be a starting point for systematic approximations such as those given in [28]. A detailed study of such aspects is deferred to later work.

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